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Intermediate Quantum Statistics for Identical Objects

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Methods to construct various algebras of creation and annihilation operators of physical objects in complex quantum state spaces with a nonnegative metric are proposed. All allowed algebras for the cases of identical nonrelativistic systems in the second quantization of the Schrodinger equation, of identical quanta of relativistic tensor fields, and of identical quanta of relativistic spinor fields are constructed. A comparison of the obtained algebras with the well-known algebras of this type (Fermi, Bose, para-Fermi, and superalgebras) is given.

1. INTRODUCTION

The mathematical formalism of a quantum theory, which deals with systems of a variable number of physical objects (particles, quanta, system, particlelike formations, and so on), has to contain the notions of creation operator $\overset{*}{a}_k$ and annihilation operator a_k for any type k of the considered objects. It is supposed that the action of these operators should be determined in space H of quantum states $|\psi\rangle$ for the considered system. Hence, the totality of the $\overset{*}{a}_k$ and a_k for $k \in \{k\}$ generates some algebra $A(\{k\})$ in the state space H, which one may call an algebra of creation and annihilation operators.

All properties of $A(\{k\})$, as well as types and statistics of the considered objects, are determined by a system of identical relations for operators $\overset{*}{a}_k$ and a_k.

The algebra $A(\{k\})$ contains operators of some physical quantities characterizing the system with a variable number of objects. In particular, the operator E of the observable total energy of such a system may be written, as a rule, in the form

$$E = \sum_k E_k, \qquad E_k = \varepsilon_k N_k \tag{1}$$

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Here ε_k is the energy of one object of type k, and $N_k \in A(\{k\})$ is the operator of a number of objects of type k, which one may consider as identical objects.

As a rule, the modern quantum theory makes use of only two standard algebras of creation and annihilation operators, which satisfy the relations

$$N_k = \overset{*}{a}_k a_k, \qquad [a_k, \overset{*}{a}_l]_{\pm} = \delta_{kl}, \qquad [a_k, a_l]_{\pm} = 0 \tag{2}$$

Here the plus sign corresponds to the Fermi algebra $A_F(\{k\})$ and the Fermi-Dirac statistics ($n_k \in \overline{0, 1}$), and the minus sign corresponds to the Bose algebra $A_B(\{k\})$ and to Bose-Einstein statistics ($n_k \in \overline{0, \infty}$).

But the development of quantum physics has introduced a set of nonstandard algebras of creation and annihilation operators which differ both from A_F and A_B. As examples of such algebras one can mention anomalous algebras (Klein, 1938; Streater and Wightman, 1961; Wigner, 1950), algebras of para-Bose operators (Wigner, 1950; Yang, 1951; Green, 1953; Greenberg and Messiah, 1965; Govorkov, 1973), algebras of para-Fermi operators (Green, 1953; Volkov, 1959; Gallindo and Indurian, 1963; Greenberg and Messiah, 1965; Govorkov, 1966, 1973; Yamada, 1968), algebras of superoperators (Roman and Aghassi, 1966; Santhanam, 1976; Kuryshkin, 1980; Madivanane and Satyanarayana, 1984), q-algebras (Aric and Coon, 1976; Kuryshkin, 1976; Siafaricas *et al.*, 1983), μ-algebras (Kuryshkin, 1976, 1980; Balashova *et al.*, 1985), ϕ-algebras (Kuryshkin, 1976; Dubkov and Kuryshkin, 1981; Grachov *et al.*, 1982), and M-algebras (Kuryshkin, 1976; Grachov *et al.*, 1982, Grachov and Kundu, 1982). One can find a brief discussion of all of these algebras in the report by Kuryshkin and Entralogo (1984).

From the point of view of the statistics for identical objects one may consider the algebras of para-Fermi operators, where

$$N_k = \tfrac{1}{2}[\overset{*}{a}_k, a_k]_- + \tfrac{1}{2}p, \qquad a_k = \sum_{j=1}^{p} b_k^{(j)} \tag{3a}$$

$$[b_k^{(j)}, \overset{*}{b}_l^{(j)}]_+ = \delta_{kl}, \qquad [b_k^{(j)}, b_l^{(j)}]_+ = = 0 \tag{3b}$$

$$[b_k^{(i)}, \overset{*}{b}_l^{(j)}]_- = [b_k^{(i)}, b_l^{(j)}]_- = 0, \qquad i \neq j \tag{3c}$$

and the algebras of superoperators, where

$$N_k = \overset{*}{a}_k a_k, \qquad [a_k, \overset{*}{a}_k]_- = 1 - \frac{p+1}{p!} \overset{*}{a}_k^p a_k^p \tag{4a}$$

$$[a_k, \overset{*}{a}_l]_- = [a_k, a_l]_- = 0, \qquad k \neq l \tag{4b}$$

are the most important algebras. In fact, for algebras (3a)-(3c) as well as (4a), (4b), p is any integer, $a_k^{p+1} = \overset{*}{a}_k^{p+1} = 0$ and $n_k \in \overline{0, p}$, i.e., the number of identical objects in the same quantum state here may be varied from 0 to p. That is why these algebras claim to some description of systems with a variable number of identical physical objects obeying some intermediate quantum statistics (between Fermi-Dirac and Bose–Einstein statistics).

In recent years the interest in the intermediate statistics, including the so-called fractional statistics, where

$$a_k\overset{*}{a}_l = e^{-i\theta_{kl}}\overset{*}{a}_l a_k, \qquad a_k a_l = e^{i\theta_{kl}} a_l a_k, \qquad k \neq l \tag{5}$$

has been increased by investigations in the theory of the quantum Hall effect (Halperin, 1984; Tao and Wu, 1985), in the general theories of field quanta (Kuryshkin, 1980; Dubkov and Kuryshkin, 1981; Grachov *et al.*, 1982; Grachov and Kundu, 1982), charged exitations (Wu, 1984; Su, 1986), monopoles (Ringwood and Woodwaard, 1984), and other quasiparticles.

This is why a look at possible algebras which may be connected with intermediate quantum statistics seems to be of interest, and it is quite natural that the main interest concerns the algebras determined in a statespace with a nonnegative metric which would allow a clear and consistent probability treatment.

2. ALGEBRAS OF CREATION AND ANNIHILATION OPERATORS FOR IDENTICAL OBJECTS

To analyze possible algebras of creation and annihilation operators for identical objects in a complex space H_+ of state vectors $|\psi\rangle$ ($\langle\psi|\psi'\rangle = \langle\psi'|\psi\rangle^*$) with a nonnegative metric ($\langle\psi|\psi\rangle \geq 0$) we start from the following intuitive definition.

Definition. The algebra $A(1)$ with one pair of generators a and $\overset{*}{a} = a^+$ mutally adjoint in H_+ is an algebra of creation and annihilation operators for some identical objects if:

(a) There exists a self-adjoint operator $N = N^+ \in A(1)$ with only integral eigenvalues from 0 to s, so that

$$N|nl_n\rangle = n|nl_n\rangle, \qquad n \in \overline{0, s}, \qquad s \geq 1, \qquad |nl_n\rangle \neq 0 \tag{6a}$$

where $|nl_n\rangle \in H_+$, and l_n are indexes of degeneration.

(b) The actions of generators a and $\overset{*}{a}$ on eigenvectors of operator N are given by the relations

$$a|0l_0\rangle = 0; \qquad a|nl_n\rangle = \alpha_{nln}|n-1, l_{p-1}\rangle, \qquad n \in \overline{1, s} \tag{6b}$$

$$\overset{*}{a}|nl_n\rangle = \beta_{nl_n}|n+1, l_{n+1}\rangle, \qquad n \in \overline{0, s-1}; \qquad \overset{*}{a}|sl_s\rangle = 0, \qquad s < \infty \tag{6c}$$

where α_{nl_n} and β_{nl_n} are nonzero complex numbers.

Relations (6) enable us to treat the vector $|nl_n> \in H_+$ as a quantum state with a fixed number n of some physical objects and operators a, $\overset{*}{a}$,

$N \in A(1)$ as, respectively, annihilation, creation, and number operators for such objects.

For finite $s \neq 1$, the number n of objects in the same quantum state may change from 0 to s, i.e., the physical objects associated with such an algebra $A(1)$ obey an intermediate statistics.

It is quite obvious that the intuitive definition (6) defines in reality a set $\{A(1)\}$ of different algebras including the Bose algebra $A_B(1)$ and the Fermi algebra $A_F(1)$. In other words, one algebra $A(1)$ may differ from another by its operator identities for the generators a and $\overset{*}{a}$, which should be added to (6).

Hence our intuitive definition (6) is not unique. Nevertheless, it makes it possible to formulate and to prove a set of theorems which brings out the unique definitions of $A(1)$ and the necessary classification of possible algebras of creation and annihilation operators for identical objects. In this paper I shall only enumerate these theorems, the proofs and immediate consequences of which are given in Kuryshkin (1987).

Theorem 1. There exists at least one subspace $H_+(s) \subset H_+$ which gives an irreducible representation of the algebra $A(1)$ with nondegenerate eigenvalues of N. Any vector $|\psi\rangle$ of $H_+(s)$ may be written as

$$|\psi\rangle = \sum_{n=0}^{s} \psi_n |n\rangle \in H_+(s) \subset H_+ \tag{7a}$$

$$N|n\rangle = n|n\rangle, \qquad \langle n|n'\rangle = \delta_{nn'}, \qquad n, n' \in \overline{0, s} \tag{7b}$$

where ψ_n are complex numbers.

Theorem 2. The algebra $A(1)$ of creation and annihilation operators in the space $H_+(s)$ is uniquely defined by a number $s \geq 1$ (*rank* of the algebra) and a set of s numbers $\lambda_n > 0$ (*parameters* of the algebra) with which the action of any $X \in A(1)$ on any $|\psi\rangle \in H_+(s)$ is uniquely determined by the relations

$$a|0\rangle = 0; \qquad a|n\rangle = \lambda_n^{1/2}|n-1\rangle, \qquad n \in \overline{1, s} \tag{8a}$$

$$\overset{*}{a}|n\rangle = \lambda_{n+1}^{1/2}|n+1\rangle, \qquad n \in \overline{0, s-1}; \qquad \overset{*}{a}|s\rangle = 0, \qquad s < \infty \tag{8b}$$

Theorem 3, In an irreducible matrix representation of $A(1)$ with $s+1$ rows and columns the representatives of a, $\overset{*}{a}$, and N may be written as

$$(a)_{kl} = \delta_{k,l-1}\lambda_k^{1/2}, \qquad (\overset{*}{a})_{kl} = \delta_{k,l+1}\lambda_{k-1}^{1/2}, \qquad (N)_{kl} = \delta_{kl}(k-1) \tag{9}$$

where $k, l \in \overline{1, s+1}$, s is the rank, and λ_n are the parameters of the algebra.

Theorem 4. The generators a and $\overset{*}{a}$ and the number operator N in an irreducible representation of $A(1)$ satisfy the operator identities

$$[N, \overset{*}{a}]_- = \overset{*}{a}, \qquad [N, a]_- = -a \tag{10}$$

$$a^{s+1} = \overset{*}{a}^{s+1} = 0, \qquad s < \infty \tag{11}$$

Theorem 5. There exists a system of $2s^2$ operator identities for the generators a and $\overset{*}{a}$ in an irreducible representation of $A(1)$, in particular, $s(s+1)/2$ identities

$$a^k \overset{*}{a}^{k+l} = \sum_{m=0}^{s-l} \mu_m^{k,k+l} \overset{*}{a}{}^{m+l} a^m \tag{12}$$

where $1 \leq k \leq k+l < s+1$, and $s(s-1)/2$ identities adjoint to (12) with $L \neq 0$. The coefficients of such identities are real and uniquely determined by the parameters λ_n of the irreducible representation with the help of recurrence relations. In particular,

$$\mu_0^{k,k+l} = \frac{\Gamma_{k+l}}{\Gamma_l}, \qquad \Gamma_0 = 1, \qquad \Gamma_n = \prod_{k=1}^{n} \lambda_k > 0 \tag{13a}$$

$$\mu_n^{k,k+l} = \frac{\Gamma_{n+k+l}}{\Gamma_n \Gamma_{n+l}} - \sum_{m=0}^{n-1} \frac{\mu_m^{k,k+l}}{\Gamma_{n-m}}, \qquad 1 \leq n < s-k-l+1 \tag{13b}$$

$$\mu_n^{k,k+l} = -\sum_{m=0}^{n-1} \frac{\mu_m^{k,k+l}}{\Gamma_{n-m}}, \qquad s-k-l+1 \leq n \leqslant s-l \tag{13c}$$

Theorem 6. Any operator x of the algebra $A(1)$ in its irreducible representation may be written in the normal form where all creation operators are on the left of all annihilation operators, i.e.,

$$x = \sum_{k,l=0}^{s} x_{kl} \overset{*}{a}{}^k a^l, \qquad x \in A(1) \tag{14}$$

where x_{kl} are complex numbers.

Theorem 7. The first s coefficients $\mu_m^{1,1}$, $m \in \overline{0, s-1}$, of the first operator identity from (12)

$$a\overset{*}{a} = \sum_{m=0}^{s} \mu_m^{1,1} \overset{*}{a}{}^m a^m \tag{15}$$

uniquely determine all parameters λ_n of $A(1)$ by the recurrence relations

$$\lambda_n = \sum_{m=0}^{n-1} \mu_m^{1,1} \frac{\Gamma_{n-1}}{\Gamma_{n-m-1}}, \qquad n \in \overline{1, s} \tag{16}$$

and for finite s the last coefficient $\mu_s^{1,1}$ of (15) is found by the equality (13c).

Theorem 8. The number operator N in an irreducible representation of $A(1)$ may be written in the normal form

$$N = \sum_{k=1}^{s} \nu_k \overset{*}{a}{}^k a^k \tag{17}$$

with real coefficients ν_k uniquely determined by the parameters λ_n with the help of the recurrence relations

$$\nu_1 = \frac{1}{\lambda_1}, \qquad \nu_n = \frac{n}{\Gamma_n} - \sum_{k=1}^{n-1} \frac{\nu_k}{\Gamma_{n-k}}, \qquad 2 \le n \le s \tag{18a}$$

On the other hand,

$$\lambda_n = n \left(\sum_{k=1}^{n} \nu_k \frac{\Gamma_{n-1}}{\Gamma_{n-k}} \right)^{-1}, \qquad n \in \overline{1, s} \tag{18b}$$

Theorem 9. The number operator N of $A(1)$ in its irreducible representation may be always written down in the form

$$N = c_0 + \sum_{k=1}^{s} c_{1,k} a^k \overset{*}{a}{}^k + \sum_{k=1}^{s} c_{2,k} \overset{*}{a}{}^k a^k \tag{19}$$

with real coefficients connected to the parameters λ_n as

$$c_0 = -\sum_{k=1}^{s} c_{1,k} \Gamma_k; \qquad s - c_0 = \sum_{k=1}^{s} c_{2,k} \frac{\Gamma_s}{\Gamma_{s-k}}, \qquad s < \infty \tag{20a}$$

$$n - c_0 = \sum_{k=1}^{s-n} c_{1,k} \frac{\Gamma_{n+k}}{\Gamma_n} + \sum_{k=1}^{n} c_{2,k} \frac{\Gamma_n}{\Gamma_{n-k}}, \qquad 1 \le n < s \tag{20b}$$

The form (19) of N is not unique.

The above theorems show that an irreducible representation of any algebra $A(1)$ of creation and annihilation operators in a complex state space H_+ with a nonnegative metric may be uniquely defined by at least three methods:

1. Rank s and parameters $\lambda_n > 0$, $n \in \overline{1, s}$, which determine the action (8) of $\overset{*}{a}$ and a on the basis (7) of subspace $H_+(s)$ are given.

2. Rank s and real numbers $\mu_m^{1,1}$, $m \in \overline{0, s-1}$, which satisfy the condition of nonnegativity of (16a) and are the first s coefficients of operator identity (15) are given.

3. Rank s and real numbers ν_k, $k \in \overline{1, s}$, which satisfy the conditions of finitary and nonegativity of (18b) and are the coefficients of N in the normal form (17) are given.

All these methods to define $A(1)$ are unique, but unfortunately are purely mathematical.

While constructing a concrete physical quantum theory for systems with a variable number of objects one usually knows only a number operator N in the form (19). Such an operator arises, for example, in the reduction of the observable energy operator to the form (1).

The operator N in the form (19) does not allow us to determine $A(1)$ uniquely, but its system of equations (20) gives a method to determine the set of allowed algebras.

I shall demonstrate the correctness of the last statement with examples of quantum field theories where

$$N = c_0 + c_1 a\overset{*}{a} + c_a \overset{*}{a} a \tag{21}$$

Here c_1 and c_2 are real coefficients determined uniquely by the quantization procedure, and c_0 is some real constant which has to be introduced in the quantum theory to eliminate the energy of the state without objects.

3. IDENTICAL NONRELATIVISTIC QUANTUM SYSTEMS

The second quantization of the nonrelativistic Schrödinger equation [the wave function $\psi(x)$ turns to be an operator $\psi(x) \in A(\{k\})$] gives the energy operator (1) with a number operator (21) for the systems in a fixed quantum state (identical systems) where $c_0 = c_1 = 0$, $c_2 = 1$.

With this one has from the system of equations (20)

$$n = \lambda_n, \qquad 1 \leq n \leq s; \qquad s = \lambda_s, \qquad s < \infty \tag{22}$$

Hence in this case any rank $s \geq 1$ is allowed, while $\lambda_n = n > 0$.

It follows now from (13) and (18a) that $\Gamma_n = n!$, $\nu_1 = 1$, $\nu_n = 0$, for $n \in \overline{1, s}$; $\mu_0^{1,1} = 1$, $\mu_1^{1,1} = -1$ for $s = 1$; and $\mu_1^{1,1} = 1$ for $s > 1$, $\mu_n^{1,1} = 0$ for $2 \leq n < s$, and $\mu_s^{1,1} = -(s+1)/\Gamma s$ for $2 \leq s < \infty$.

Thus, for a quantum theory of systems with a variable number of identical nonrelativistic quantum systems we have obtained the following results:

(a) An algebra $A(1)$ of infinite rank in a space $H_+(\infty)$ is allowed. For this algebra $\lambda_n = n$, $n \in \overline{1, \infty}$,

$$N = \overset{*}{a} a, \qquad a^n \neq 0, \qquad a\overset{*}{a} = 1 + \overset{*}{a} a \tag{23}$$

Obviously this is Bose Algebra, $A_B(1)$.

(b) Algebras of any finite rank $s \geq 1$ in an $(s+1)$-dimensional space $H_+(s)$ are allowed. For these algebras $\lambda_n = n$, $n \in \overline{1, s}$,

$$N = \overset{*}{a} a, \qquad a^{s+1} = 0, \qquad a\overset{*}{a} = 1 + \overset{*}{a} a - \frac{s+1}{s!} \overset{*}{a}{}^s a^s \tag{24}$$

The last algebras coincide with the algebras of superoperators (4) for identical objects. In the case $s = 1$ this is obviously the Fermi algebra $A_F(1)$.

One can obtain the system of operator identities (12) for algebras (24) by determining the coefficients (13) with $\lambda_n = n$. As an example, let us write these identities and the matrix representation for the algebra (24) of rank 2: $\{|n>\} = |0\rangle, |1\rangle, |2\rangle$; $N = \overset{*}{a}a$,

$$a^3 = 0, \qquad a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$a\overset{*}{a} = 1 + \overset{*}{a}a - \tfrac{3}{2}\overset{*}{a}{}^2a^2, \qquad a\overset{*}{a}{}^2 = 2\overset{*}{a} - 2\overset{*}{a}{}^2a, \qquad a^2\overset{*}{a}{}^2 = 2 - 2\overset{*}{a}a + \overset{*}{a}{}^2a^2$$

and the adjoint expressions.

4. IDENTICAL QUANTA OF RELATIVISTIC TENSOR FIELDS

A quantization of a tensor field gives the observable energy operator (1) with a number operator (21) where $c_1 = c_2 = 1/2$.

In this case we obtain from (20)

$$\lambda_1 = -c_0; \qquad \lambda_s = 2s - 2c_0, \qquad s < \infty \tag{25a}$$

$$\lambda_{n+1} = -\lambda_n + 2n - 2c_0, \qquad 1 \leq n < s \tag{25b}$$

If $s = 2s_1$, where $s_1 \leq 1$ is an integer from (25), we have $\lambda_s = 2s_1$ and at the same time $\lambda_s = 4s_1 - 2c_0$, i.e., a contradiction, $\lambda_1 = -2s_1 < 0$. If $s = 2s_1 + 1$, we have from (25) $\lambda_s = 2s_1 - 2c_0$ and at the same time $\lambda_s = 4s_1 + 2 - 2c_0$, i.e., a contradiction, $s_1 = -1$. If $s = \infty$, the system (25) has the parametric solution

$$\lambda_{2k} = 2k, \qquad \lambda_{2k+1} = 2k + c, \qquad c > 0 \tag{26}$$

Hence, for a quantum theory of systems with a variable number of identical quanta of a tensor field we have obtained the following results:

(a) Algebras $A(1)$ of infinite rank in a space $H_+(\infty)$ are allowed. For these algebras λ_n are given by (26) and

$$N = \tfrac{1}{2}[\overset{*}{a}, a]_+ - \tfrac{1}{2}c, \qquad a^m \neq 0 \tag{27a}$$

$$a\overset{*}{a} = c + \frac{2-c}{c}\overset{*}{a}a + \frac{2(c-1)}{c^2}\overset{*}{a}{}^2a^2 + \cdots \tag{27b}$$

The operator identity (27b) of these algebras contains an infinite number of items [the coefficients are determined by (13) and (26)] except for the case $c = 1$, when (27) is a Bose algebra $A_B(1)$.

(b) Algebras $A(1)$ of finite ranks are not allowed.

5. IDENTICAL QUANTA OF RELATIVISTIC SPINOR FIELDS

A quantization of a spinor field gives the observable energy operator (1) with a number operator (21) where $c_1 = -c_2 = -\frac{1}{2}$.

With this, the system of equations (20) gives

$$\lambda_1 = 2c_0; \qquad \lambda_s = 2s - 2c_0, \qquad s < \infty \tag{28a}$$

$$\lambda_{n+1} = \lambda_n + 2c_0 - 2n, \qquad 1 \leq n < s \tag{28b}$$

If $s = \infty$, we have $\lambda_n = 2nc_0 - n(n-1)$, i.e., a contradiction, $\lambda_n < 0$ for large enough n. If s is any integer, we have $\lambda_s = 2sc_0 - s(s-1)$ and at the same time $\lambda_s = 2s - 2c_0$, i.e., $2c_0 = s$ and definitively

$$c_0 = s/2, \qquad \lambda_n = n(s-n+1) > 0, \qquad n \in \overline{1, s} \tag{29}$$

Thus, for a quantum theory of systems with a variable number of identical quanta of a spinor field we have obtained:

(a) Algebras $A(1)$ of the infinite rank are not allowed.

(b) Algebras $A(1)$ of any finite rank $s \geq 1$ in an $(s+1)$-dimensional space $H_+(s)$ are allowed. For these algebras one has $\lambda_n = n(s-n+1)$, $n \in \overline{1, s}$, and

$$N = \tfrac{1}{2}[\overset{*}{a}, a]_- + \tfrac{1}{2}s, \qquad a^{s+1} = 0 \tag{30a}$$

$$a\overset{*}{a} = s + \frac{s-2}{s}\overset{*}{a}a + \cdots \tag{30b}$$

The operator identity (30b) contains all the items $\overset{*}{a}{}^k a^k$, $k \in \overline{1, s}$, with the coefficients determined by (13) and (29). The particular case $s = 1$ of (30) is obviously the Fermi algebra $A_F(1)$.

As an example, let us write the main relations for algebra (30) of rank 2: $\{|n\rangle\} = |0\rangle, |1\rangle$; $N = \overset{*}{a}a/2 + \overset{*}{a}{}^2 a^2/4$ and

$$a^3 = 0, \qquad a = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$a\overset{*}{a} = 2 - \tfrac{1}{2}\overset{*}{a}{}^2 a^2, \qquad a\overset{*}{a}{}^2 = 2\overset{*}{a} - \overset{*}{a}{}^2 a, \qquad a^2\overset{*}{a}{}^2 = 4 - 2\overset{*}{a}a$$

and the adjoint expressions.

6. CONCLUDING REMARKS

It is necessary to mention one more type of algebra which may be also connected with intermediate statistics for identical objects and seems to be useful in the quantum theory of identical spin waves. For such algebras the number operator has the form (21) with $c_1 = -1$ and $c_2 = 0$. The system of equations (20) in this case forbids the infinite rank, but allows any finite rank $s \geq 1$ with $c_0 = s$, $\lambda_n = s - n + 1$, and

$$N = s - a\overset{*}{a}, \qquad a^{s+1} = 0 \tag{31a}$$

$$a\overset{*}{a} = s - \frac{1}{s}\overset{*}{a}a + \cdots \tag{31b}$$

The coefficients of the items $\overset{*}{a}{}^k a^k$, $k \in \overline{2, s}$, in (31b) are given by (13).

The particular case $s = 1$ of the algebra (31) is naturally $A_F(1)$. The main relations for algebra (31) of rank 2 are: $\{|n\rangle\} = |0\rangle, |1\rangle, |2\rangle$; $N = \overset{*}{a}a/2 + 3\overset{*}{a}{}^2 a^2/4$,

$$a^3 = 0, \qquad a = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a\overset{*}{a} = 2 - \tfrac{1}{2}\overset{*}{a}a - \tfrac{3}{8}\overset{*}{a}{}^2 a^2, \qquad a\overset{*}{a}{}^2 = \overset{*}{a} - \tfrac{1}{2}\overset{*}{a}{}^2 a, \qquad a^2\overset{*}{a}{}^2 = 2 - \overset{*}{a}a - \tfrac{1}{2}\overset{*}{a}a - \tfrac{1}{2}\overset{*}{a}{}^2 a^2$$

and the adjoint expressions.

We have already indicated the Bose algebra [(23) and a particular case of (27)] and Fermi algebra [particular cases of (24), (30), and (31)]. We have also mentioned that the algebras (4) of superoperators for identical objects completely coincide with algebras (24) for identical nonrelativistic quantum systems.

As far as the algebras (3) of para-Fermi operators are concerned, we have to underline that irreducible representations of all allowed algebras $A(1)$ with the number operator [compare with (3a)] $N = [\overset{*}{a}, a]_-/2 + c_0$ are given by relations (30). It is quite easy to show that a representation of a and $\overset{*}{a}$ by the Green ansatz (3a) is compatible with (30a) but is incompatible with (30a) and (30b) jointly if $s \geq 2$. Hence, the para-Fermi algebras (3) are not algebras of creation and annihilation operators for identical objects at least in the sense of the definition (6). This fact demands a special and careful investigation.

Finally, we have to conclude that objects obeying intermediate statistics (if they exist) might be described with the help of algebras of types (24), (30), and (31), so that intermediate statistics for identical nonrelativistic quantum systems are connected with algebras (24), intermediate statistics for identical quanta of tensor fields do not exist, and intermediate statistics for identical quanta of spinor fields are connected with algebras (30).

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